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The relativistic many-body problem and application to bottomonium

Marcos Moshinsky and Verónica Riquer

Instituto de Física, UNAM Apartado Postal 20-364, 01000 México, DF, Mexico

E-mail: moshi@fisica.unam.mx

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Abstract

It is well known that the analysis of a relativistic n-body problem invariant under the transformations of the Poincaré group and involving only one time has only been done for $n = 1$. For $n > 1$ one uses the second quantization formalism of field theory. In this paper we state it in the ordinary space time coordinates associated with the n-bodies as Dirac did in the one body case. We apply the formalism first to the two-body problem and have a development of the Hamiltonian in terms of powers of $(1/c^2)$, that allows us to determine the spectra of bottomonium and compare with the experimental results.

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1. Introduction

In non-relativistic quantum mechanics the passage from the single-body to a many-body problem, for non-interacting particles, is a trivial summation of the single-body Hamiltonians.

In the relativistic case if we want to use a similar procedure and keep the problem invariant under Poincaré transformations, we have to deal with the times associated with each body.

In this paper we first show that for a system of n-particles with the same mass and spin $(1/2)$ we can, with the help of appropriate matrices, formulate the many-body problem, in the centre-of-mass reference frame, with the appearance of only one time.

While our procedure applies to arbitrary n-body problems, the algebra increases as a function of n. Thus, the simplest problem we can attack is the two-body problem for which we give a procedure to determine its spectra starting from its non-relativistic formulation.

The particular case to which we have applied our formalism is the quark–antiquark system of the bottom type for which an interaction potential is given in the literature. We discuss first a variational procedure from which we determine the non-relativistic eigenvalues and eigenstates and use the latter to determine the relativistic correction to order $(1/c^2)$.

We finally state the problem when the two masses are different and indicate possible applications to other systems.

2. A formulation of the relativistic many-body problem

A many-body problem usually starts with the case of non-interacting particles and, for simplicity, of the same mass.

In the non-relativistic case the wave equation can be written as

$$\left(\frac{1}{2m}\sum_s p_s^2\right)\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}. \tag{1}$$

In the relativistic case, and taking particles of spin $\frac{1}{2}$, an obvious generalization from the Dirac equations would be

$$\sum_{s=1}^{n}\left(\gamma_s^\mu p_{\mu s}+1\right)\psi = 0 \tag{2}$$

where repeated indices μ are summed with respect to the values $\mu = 0, 1, 2, 3$ and the index $\mu = 0$ refers to the time, and where we use the units $\hbar = m = c = 1$. The $p_{\mu s}$ is a covariant energy–momentum four vector and γ_s^μ are the contravariant matrices related to

$$\gamma_s^0 = \beta_s \qquad \gamma_s^i = \beta_s\alpha_{is} \qquad i = 1, 2, 3 \qquad s = 1, 2, \ldots, n \tag{3}$$

with β and $\boldsymbol{\alpha}$ having the usual definitions [1].

Equation (2) is certainly an invariant of the Poincaré group but is not satisfactory because it introduces n times through $p_{0s} = -\mathrm{i}\partial/\partial x_{0s}$.

How can we find a many-body problem, still invariant under the Poincaré group but, in an appropriate system of reference, involving only one time [2, 3]?

We start denoting by u_μ a unit time-like 4-vector which means that there is a reference frame in which it takes the form

$$(u_\mu) = (1, 0, 0, 0). \tag{4}$$

With the help of the 4-vector (4), we can define the Lorentz scalars [3]

$$\Gamma = \prod_{r=1}^{n}\left(\gamma_r^\mu u_\mu\right) \qquad \Gamma_s = \left(\gamma_s^\mu u_\mu\right)^{-1}\Gamma \tag{5}$$

where $\left(\gamma_s^\mu u_\mu\right)^{-1}$ eliminates the corresponding term in Γ and Γ_s is still in product form.

We now propose that instead of equation (2) we have the Lorentz invariant one [3]

$$\sum_{s=1}^{n}\Gamma_s\left(\gamma_s^\mu p_{\mu s}+1\right)\psi = 0. \tag{6}$$

With the help of the total energy–momentum 4-vector

$$P_\mu = \sum_{s=1}^{n} p_{\mu s} \qquad \mu = 0, 1, 2, 3 \tag{7}$$

we show that, in the frame of reference where $(u_\mu) = (1, 0, 0, 0)$, equation (6) takes the form

$$\left[\Gamma^0\sum_{s=1}^{n} p_{0s} + \sum_{s=1}^{n}\Gamma_s^0(\boldsymbol{\gamma}_s\cdot\mathbf{p}_s+1)\right]\psi = 0 \tag{8}$$

where boldface letters mean three-dimensional vectors and

$$\Gamma^0 \equiv \prod_{r=1}^{n}\gamma_r^0 \qquad \Gamma_s^0 \equiv \left(\gamma_s^0\right)^{-1}\Gamma^0. \tag{9}$$

Multiplying equation (8) by Γ^0 and using equations (3), (7) and (9) we obtain

$$\left[-P^0 + \sum_{s=1}^{n}(\boldsymbol{\alpha}_s \cdot \mathbf{p}_s + \beta_s)\right]\psi = 0 \tag{10}$$

where we used a metric in which $P_0 = -P^0$ and the latter is the zero component of P^μ, i.e. the total energy of the system.

We would like equation (6) to represent the system of particles where the centre of mass is at rest and this can be achieved if we define

$$u_\mu = P_\mu(-P_\tau P^\tau)^{-\frac{1}{2}} \tag{11}$$

as when $P_i = 0, i = 1, 2, 3$, we have $u_i = 0, u_0 = 1$.

For interactions depending on the relative coordinates

$$x_\mu^{st} \equiv x_{\mu s} - x_{\mu t} \tag{12}$$

we can define

$$x_{\perp\mu}^{st} \equiv x_\mu^{st} - \left(x_\tau^{st} u^\tau\right) u_\mu \tag{13}$$

and thus suppressing the indices s, t we have that

$$r^2 \equiv \left(x_{\perp\mu} x_\perp^\mu\right) \tag{14}$$

is a Poincaré invariant and this is also true for any function of it.

We wish to end this section by indicating that there are several ways in which the relativistic many-body problem can be formulated. As an example we mention the paper of J Carbonell *et al* on 'Explicit covariant light-front dynamics and relativistic few-body systems' given in [4].

3. The Hamiltonian of the two-body problem

We shall consider our equations in the centre-of-mass frame of reference and use primes for cgs units to get

$$[(c'\boldsymbol{\alpha}_1 \cdot \mathbf{p}_1' + m'c'^2\beta_1) + (c'\boldsymbol{\alpha}_2 \cdot \mathbf{p}_2' + m'c'^2\beta_2) + V(r')]\Psi = E'\Psi \tag{15}$$

where for the potential we use one frequently proposed for the quark–antiquark system of the form [5]

$$V(r') = q'r' - \frac{b'^2}{r'} \tag{16}$$

where the term $(-b'^2/r')$ is due to a one gluon exchange (similar to the Coulomb interaction for the one photon exchange of the electron–positron system) while $q'r'$ is the potential that provides the confinement. As in the centre-of-mass frame the total momentum $\mathbf{P}' = \mathbf{p}_1' + \mathbf{p}_2' = 0$ we can write $\mathbf{p}_1' = -\mathbf{p}_2' = \mathbf{p}'$ with $\mathbf{p}' = \frac{1}{2}(\mathbf{p}_1' - \mathbf{p}_2')$ so equation (15) becomes

$$[c(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \mathbf{p} + 2c^2(\beta_1 + \beta_2) + V(r)]\Psi = E\Psi \tag{17}$$

where we used units in which $\mu' \equiv (m'/2), \hbar$ and b' equal to 1 so we have

$$c = \frac{\hbar c'}{b'^2} \qquad V(r) = qr - \frac{1}{r} \qquad q = \frac{q'\hbar^4}{\mu'^2 b'^6} \tag{18}$$

where c is the velocity of light in the units we are now using.

As we are discussing a two-body problem $\alpha_1, \alpha_2, \beta_1, \beta_2$ have the form

$$\begin{aligned} \alpha_1 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \\ \beta_1 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \beta_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \end{aligned} \tag{19}$$

Introducing these direct products explicitly in equation (17) we obtain

$$\begin{aligned} \mathcal{O}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \equiv \Bigg\{ & 2c \begin{bmatrix} 0 & \mathbf{s}_1 \cdot \mathbf{p} & -\mathbf{s}_2 \cdot \mathbf{p} & 0 \\ \mathbf{s}_1 \cdot \mathbf{p} & 0 & 0 & -\mathbf{s}_2 \cdot \mathbf{p} \\ -\mathbf{s}_2 \cdot \mathbf{p} & 0 & 0 & \mathbf{s}_1 \cdot \mathbf{p} \\ 0 & -\mathbf{s}_2 \cdot \mathbf{p} & \mathbf{s}_1 \cdot \mathbf{p} & 0 \end{bmatrix} \\ & + 4c^2 \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} - [E - V(r)] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Bigg\} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = 0 \end{aligned} \tag{20}$$

where we have replaced $\sigma_t, t = 1$ or 2 by spin matrices

$$\mathbf{s}_1 = \tfrac{1}{2}\sigma_1 \qquad \mathbf{s}_2 = \tfrac{1}{2}\sigma_2. \tag{21}$$

4. The second-order equations for the two-body problem

Equation (20) can be written in the form

$$2c \begin{pmatrix} \mathbf{s}_1 \cdot \mathbf{p} & -\mathbf{s}_2 \cdot \mathbf{p} \\ -\mathbf{s}_2 \cdot \mathbf{p} & \mathbf{s}_1 \cdot \mathbf{p} \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} E - V(r) - 4c^2 & 0 \\ 0 & E - V(r) + 4c^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_4 \end{pmatrix} \tag{22}$$

$$2c \begin{pmatrix} \mathbf{s}_1 \cdot \mathbf{p} & -\mathbf{s}_2 \cdot \mathbf{p} \\ -\mathbf{s}_2 \cdot \mathbf{p} & \mathbf{s}_1 \cdot \mathbf{p} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_4 \end{pmatrix} = [E - V(r)] \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} \tag{23}$$

Dividing equation (23) by $[E - V(r)]$ and substituting in equation (22), we obtain with the use of the commutator

$$[(\mathbf{s}_t \cdot \mathbf{p}), (E - V)^{-1}] = \frac{1}{\mathrm{i}}(E - V)^{-2}(\mathbf{s}_t \cdot \mathbf{r}) \frac{1}{r} \frac{\mathrm{d}V}{\mathrm{d}r} \tag{24}$$

and the properties of the Pauli matrices

$$\sigma_i \sigma_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\sigma_k \tag{25}$$

that equations (22) and (23) reduce to

$$\begin{aligned} & \left[\frac{1}{\mathrm{i}} 4c^2 (E - V)^{-2} \frac{1}{r} \frac{\mathrm{d}V}{\mathrm{d}r} \right] \\ & \qquad \times \begin{bmatrix} \frac{1}{2}(\mathbf{r} \cdot \mathbf{p}) + \frac{\mathrm{i}}{2}(\mathbf{S} \cdot \mathbf{L}) & \frac{1}{2}(\mathbf{r} \cdot \mathbf{p}) + \frac{\mathrm{i}}{2}(\mathbf{S} \cdot \mathbf{L}) - (\mathbf{S} \cdot \mathbf{r})(\mathbf{S} \cdot \mathbf{p}) \\ \frac{1}{2}(\mathbf{r} \cdot \mathbf{p}) + \frac{\mathrm{i}}{2}(\mathbf{S} \cdot \mathbf{L}) - (\mathbf{S} \cdot \mathbf{r})(\mathbf{S} \cdot \mathbf{p}) & \frac{1}{2}(\mathbf{r} \cdot \mathbf{p}) + \frac{\mathrm{i}}{2}(\mathbf{S} \cdot \mathbf{L}) \end{bmatrix} \\ & \qquad \times \begin{bmatrix} \psi_1 \\ \psi_4 \end{bmatrix} + \frac{4c^2}{(E - V)} \begin{bmatrix} \frac{1}{2}p^2 & \frac{1}{2}p^2 - (\mathbf{S} \cdot \mathbf{p})^2 \\ \frac{1}{2}p^2 - (\mathbf{S} \cdot \mathbf{p})^2 & \frac{1}{2}p^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_4 \end{bmatrix} \\ & \qquad = \begin{bmatrix} E - V - 4c^2 & 0 \\ 0 & E - V + 4c^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_4 \end{bmatrix} \end{aligned} \tag{26}$$

where $\mathbf{S} = (\mathbf{s}_1 + \mathbf{s}_2)$ and $\mathbf{L} = (\mathbf{r} \times \mathbf{p})$ are, respectively, the total spin and total orbital angular momentum.

5. The relativistic correction to order $(1/c^2)$

We note first that in cgs units the rest energy of our system is $2m'c'^2 = 4\mu'c'^2$ where μ' is the reduced mass. In our units $\hbar = \mu' = b' = 1$ the rest energy is $4c^2$ and the total energy is then

$$E = 4c^2 + \epsilon \tag{27}$$

with ϵ being the binding energy we want to determine.

We now introduce the short hand notation

$$a \equiv \frac{1}{2}(\mathbf{r}\cdot\mathbf{p}) + \frac{\mathrm{i}}{2}(\mathbf{S}\cdot\mathbf{L}) \qquad b \equiv \frac{1}{2}(\mathbf{r}\cdot\mathbf{p}) + \frac{\mathrm{i}}{2}(\mathbf{S}\cdot\mathbf{L}) - (\mathbf{S}\cdot\mathbf{r})(\mathbf{S}\cdot\mathbf{p}) \tag{28}$$

$$u \equiv \frac{1}{2}p^2 \qquad v \equiv \frac{1}{2}p^2 - (\mathbf{S}\cdot\mathbf{p})^2 \tag{29}$$

$$w \equiv [E - V(r)] = [4c^2 + \epsilon - V(r)] \tag{30}$$

which allow us to write equation (26) as

$$\left\{-4\mathrm{i}c^2 w^{-2} r^{-1}(\mathrm{d}V/\mathrm{d}r)\begin{bmatrix} a & b \\ b & a \end{bmatrix} + 4c^2 w^{-1}\begin{bmatrix} u & v \\ v & u \end{bmatrix}\right\}\begin{pmatrix}\psi_1 \\ \psi_4\end{pmatrix}$$
$$= \left\{w\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} - 4c^2\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\right\}\begin{pmatrix}\psi_1 \\ \psi_4\end{pmatrix}. \tag{31}$$

The left-hand side of equation (31) can now be diagonalized with the help of the matrices

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix} \qquad U^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix} \tag{32}$$

as

$$U^{-1}\begin{bmatrix} a & b \\ b & a \end{bmatrix}U = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} \qquad U^{-1}\begin{bmatrix} u & v \\ v & u \end{bmatrix}U = \begin{bmatrix} u+v & 0 \\ 0 & u-v \end{bmatrix} \tag{33}$$

while on the right-hand side the similarity transformation has no effect on the unit matrix but we have

$$U^{-1}\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}U = -\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}. \tag{34}$$

Thus writing

$$\begin{pmatrix}\psi_1 \\ \psi_4\end{pmatrix} = U\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} \tag{35}$$

and multiplying equation (31) by U^{-1} we get the pair of equations

$$\left\{-4\mathrm{i}c^2 w^{-2}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(a+b) + 4c^2 w^{-1}(u+v) - w\right\}\psi_+ = 4c^2\psi_- \tag{36}$$

$$\left\{-4\mathrm{i}c^2 w^{-2}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(a-b) + 4c^2 w^{-1}(u-v) - w\right\}\psi_- = 4c^2\psi_+. \tag{37}$$

Using equation (36) to express ψ_- in terms of ψ_+ and substituting in equation (37) we get

$$\left\{\left[-4\mathrm{i}c^2 w^{-2}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(a-b) + 4c^2 w^{-1}(u-v) - w\right]\right.$$
$$\left.\times\left[-4\mathrm{i}c^2 w^{-2}\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(a+b) + 4c^2 w^{-1}(u+v) - w\right]\right\}\psi_+ = 16c^4\psi_+. \tag{38}$$

We note that a, b, u, v do not depend on c but w does, and negative powers of it can be developed in a series of inverse powers of c^2 as

$$4c^2w^{-1} = 1 + \sum_{m=1}^{\infty}(-1)^m\left[\frac{\epsilon - V(r)}{4c^2}\right]^m \tag{39}$$

$$4c^2w^{-2} = -\sum_{m=1}^{\infty}(-1)^m\frac{m[\epsilon - V(r)]^{m-1}}{(4c^2)^m}. \tag{40}$$

Keeping only the $(1/c^2)$ term we finally get

$$H\psi_+ \equiv \left\{\left[\frac{p^2}{2} + V(r)\right] - \frac{1}{8c^2}\left[\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(\mathbf{S}\cdot\mathbf{L}) + (\mathbf{S}\cdot\mathbf{p})^2[p^2 - (\mathbf{S}\cdot\mathbf{p})^2]\right] - \frac{p^4}{32c^2}\right\}\psi_+ = \epsilon\psi_+. \tag{41}$$

We note that the Hamiltonian in equation (41) has the term $(\mathbf{S}\cdot\mathbf{p})^2[p^2 - (\mathbf{S}\cdot\mathbf{p})^2]$, which vanishes as we shall proceed to show.

We can take the third direction of our reference frame parallel to that of the vector $\mathbf{p}$ and in this case

$$(\mathbf{S}\cdot\mathbf{p})^2[p^2 - (\mathbf{S}\cdot\mathbf{p})^2] = p^2S_3^2\left[p^2 - S_3^2p^2\right] = p^4S_3^2\left[\mathbf{1} - S_3^2\right] \tag{42}$$

and as we also have that for $s = 0$, $S_3 = 0$ while for $s = 1$ we obtain

$$S_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad \mathbf{1} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{43}$$

we see that

$$S_3^2\left[\mathbf{1} - S_3^2\right] = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} = \mathbf{0}. \tag{44}$$

Thus equation (41) reduces to

$$H\psi_+ = \left\{\left[\frac{p^2}{2} + V(r)\right] - \frac{1}{8c^2}\left[\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(\mathbf{S}\cdot\mathbf{L})\right] - \frac{p^4}{32c^2}\right\}\psi_+ = \epsilon\psi_+. \tag{45}$$

Note the difference between this problem and the corresponding equation for the hydrogen atom in atomic units that we write below [6]

$$H_h\Psi \equiv \left\{\left[\frac{1}{2}p^2 - \frac{1}{r}\right] + \frac{1}{c^2}\left[-\frac{p^4}{8} - \frac{1}{4r^2}\frac{\partial}{\partial r} + \frac{1}{2r^3}(\mathbf{L}\cdot\mathbf{S})\right]\right\}\Psi = \epsilon\Psi. \tag{46}$$

The spins in equation (45) are $s = 0, 1$ while in equation (46) they are $s = \frac{1}{2}$.

6. The binding energies of the quark–antiquark system

We need first to obtain eigenstates of the non-relativistic part of equation (45)

$$H^q\psi = \left(\frac{1}{2}p^2 + qr - \frac{1}{r}\right)\psi = \epsilon\psi. \tag{47}$$

This will be done variationally employing the discrete basis of Sturm–Coulomb states. Having then the eigenvalues and eigenstates of equation (47) we use the latter in first-order

Table 1. Experimental spectra of bottomonium and our non-relativistic results (in GeV).

	Experimental	Theoretical
$l = 0,\ j = 1$		
$\Upsilon(2S) - \Upsilon(1S)$	0.562 96	0.573 621
$\Upsilon(3S) - \Upsilon(1S)$	0.894 9	0.928 582
$\Upsilon(4S) - \Upsilon(1S)$	1.119 7	1.217 7
$\Upsilon(10\,860) - \Upsilon(1S)$	1.404 7	1.471 97
$\Upsilon(11\,020) - \Upsilon(1S)$	1.558 7	1.703 66
$l = 1,\ j = 0$		
$\chi_{bo}(1P) - \Upsilon(1S)$	0.399 6	0.467 9509
$\chi_{bo}(2P) - \Upsilon(1S)$	0.771 8	0.833 7729
$l = 1,\ j = 1$		
$\chi_{b1}(1P) - \Upsilon(1S)$	0.432 4	0.467 9509
$\chi_{b1}(2P) - \Upsilon(1S)$	0.794 9	0.833 7729
$l = 1,\ j = 2$		
$\chi_{b2}(1p) - \Upsilon(1S)$	0.452 3	0.467 9509
$\chi_{b2}(2p) - \Upsilon(1S)$	0.808 2	0.833 7729

perturbation to get the contribution up to $(1/c^2)$ terms of the relativistic correction. For details of this calculation you can look at the appendix of this paper.

In the case of bottomonium we use the potential of equation (16) with the values given in the literature [5]

$$q' = \frac{\sigma}{\hbar c'} \qquad \sigma = 0.182\,(\mathrm{GeV})^2 \tag{48}$$

$$b'^2 = \frac{4}{3}\alpha_s \hbar c' \qquad \text{with } \alpha_s = 0.39 \quad \text{so that} \quad b'^2 = (0.52)\hbar c' \tag{49}$$

$$\mu' = 2.1\,\mathrm{GeV}. \tag{50}$$

The only parameter in equation (47) is q defined by equation (18) and which takes the value

$$q = \frac{q'\hbar^4}{\mu'^2 b'^6} = 0.2935. \tag{51}$$

In figure 1 we give the energy levels of the non-relativistic problem compared with experiment [7]. The agreement is reasonable as we see numerically in table 1.

The relativistic correction calculated as indicated above gives only a small value and always in the direction that approximates the experimental result as shown in table 2.

7. The Hamiltonian of the two-body problem when the masses are different

The analysis of this case is very similar to that which we carried in section 3 when the masses were equal. Thus, we only present the result

$$H = \left(\frac{p^2}{2\mu} + V\right) - \frac{1}{2m_1^2c^2}\left[\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(\mathbf{s}_1 \cdot \mathbf{L})\right] - \frac{1}{2m_2^2c^2}\left[\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}(\mathbf{s}_2 \cdot \mathbf{L})\right] - \frac{p^4}{8\mu c^2}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{M_+\mu}\right) \tag{52}$$

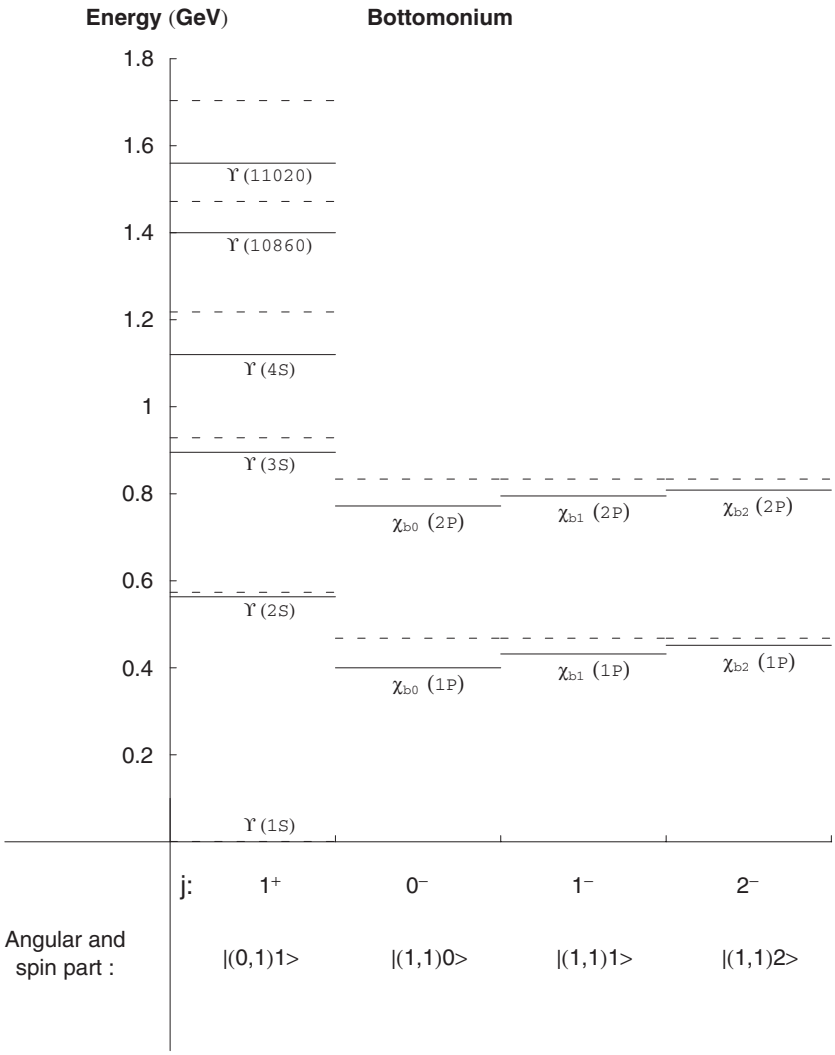

Figure 1. We give using full lines the experimental energy levels of bottomonium putting in the abscissa their total angular momentum. With dashed lines we give the non-relativistic theoretical results and indicate below the abscissa the ket $|(l, s)j\rangle$ with l, s, j being, respectively, the orbital, spin and total angular momentum. Below each experimental level we indicate the orbital angular momentum in spectroscopic notation. The spin is always 1. We give only the excitation, i.e. the energy of the level minus the energy of $\Upsilon(1S)$, so the experimental and theoretical results coincide at zero value.

where we still use units in which $\mu' = b'^2 = \hbar = 1$ and the masses m_1, m_2 of the two particles are given by

$$m_1 = \frac{m_1'}{\mu'} \qquad m_2 = \frac{m_2'}{\mu'} \qquad M_+ = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{53}$$

Equation (52) reduces to equation (45) when $m_1 = m_2 = 2$, $\mu = 1$, $M_+ = 4$.

We plan to apply equation (52) to further calculations involving quark–antiquarks of different masses.

Table 2. Experimental spectra of (Υ) states of bottomonium and our relativistic corrections (in GeV).

	Experimental	Theoretical	Relativistic correction
$l = 0,\ j = 1$			
$\Upsilon(2S) - \Upsilon(1S)$	0.562 96	0.573 621	−0.001 879 27
$\Upsilon(3S) - \Upsilon(1S)$	0.894 9	0.928 582	−0.000 821 763
$\Upsilon(4S) - \Upsilon(1S)$	1.119 7	1.217 7	−0.000 487 554
$\Upsilon(10\,860) - \Upsilon(1S)$	1.404 7	1.471 97	−0.000 343 503
$\Upsilon(11\,020) - \Upsilon(1S)$	1.558 7	1.703 66	−0.000 271 208

8. Conclusions

We developed a formalism that is invariant under the Poincaré group for the n-body system, and allows an expansion in powers of $(1/c^2)$ of the Hamiltonian starting with the non-relativistic ones. We discuss in detail the situation when $n = 2$, and apply our results to derive the spectrum of bottomonium and compare it with the experimental data.

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Appendix. Eigenvalues and eigenstates of the non-relativistic problem

Our first objective will be to obtain solutions of equation (47) and, in particular the values of ϵ, using a Ritz variational method. For this purpose we need a complete set of solutions of a quantum mechanical problem that can serve as the variational basis. An obvious basis is that of a Coulomb problem but this is not complete if we use only discrete levels, as there is also a contribution from states in the continuum. Thus, we turn to harmonic oscillator states but these converge slowly for the Hamiltonian of equation (45). We then decided on the Sturm–Coulomb states $R_{nl}(r)$ which satisfy the equation [8]:

$$p^2 R_{nl}(r) = \left(\frac{\nu}{r} - \frac{1}{4}\right) R_{nl}(r) \tag{A.1}$$

with

$$\nu = n + l + 1 \qquad p^2 = -\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{r^2}. \tag{A.2}$$

The functions $R_{nl}(r)$ then take the form

$$R_{nl}(r) = \sqrt{\frac{n!}{(n+2l+1)!}}\, r^l\, \mathrm{e}^{-r/2} L_n^{2l+1}(r) \tag{A.3}$$

where L_n^{2l+1} is a Laguerre polynomial.

These states are orthonormal in a differential radial volume $r\,\mathrm{d}r$ but not in the physical one $r^2\,\mathrm{d}r$ so we have

$$\int_0^\infty R_{nl}(r) R_{n'l}(r) r^2\,\mathrm{d}r \equiv \langle n'l|nl\rangle = b(n'l, nl, 2) \tag{A.4}$$

with the b defined below.

In equation (A.1) we made a definite choice of parameters for the $R_{nl}(r)$ and if we want to continue using them we can introduce the parameter in the H^q of equation (47) by the substitution [8]

$$\mathbf{p} \to \lambda \mathbf{p} \qquad \mathbf{r} \to \frac{\mathbf{r}}{\lambda} \tag{A.5}$$

so we have as our Hamiltonian

$$H^q = \frac{1}{2}\lambda^2 p^2 + \frac{qr}{\lambda} - \frac{\lambda}{r}. \tag{A.6}$$

For the variational Ritz method we need Hamiltonian matrices (to be denoted by a bold face capital letter) whose elements are calculated with respect to the states $R_{nl}(r)$, i.e.

$$\mathbf{H}^{ql} = \left\| \int_0^\infty R_{n'l}(r) \left[\left(\frac{1}{2}\lambda^2 p^2 + \frac{qr}{\lambda} - \frac{\lambda}{r} \right) R_{nl}(r) \right] r^2 \, \mathrm{d}r \right\|. \tag{A.7}$$

Making use of equation (A.7), where we introduce the upper index l in H^q, as the orbital angular momentum is an integral of motion and thus the matrix representation is characterized by its eigenvalue $l(l+1)$, and we note that p^2 is given by equation (A.2),

$$\begin{aligned} \mathbf{H}^{ql} &= \left\| \int_0^\infty R_{n'l}(r) \left\{ \left[\frac{1}{2}\lambda^2 \left(\frac{\nu}{r} - \frac{1}{4} \right) + \frac{qr}{\lambda} - \frac{\lambda}{r} \right] R_{nl}(r) \right\} r^2 \, \mathrm{d}r \right\| \\ &= \left\| \left(\frac{1}{2}\lambda^2 \nu - \lambda \right) b(n'l, nl, 1) - \frac{1}{8}\lambda^2 b(n'l, nl, 2) + \frac{q}{\lambda} b(n'l, nl, 3) \right\| \end{aligned} \tag{A.8}$$

where

$$b(n'l, nl, k) \equiv \int_0^\infty R_{n'l}(r) R_{nl}(r) r^k \, \mathrm{d}r \tag{A.9}$$

that was calculated explicitly in [9].

Our first objective is to determine the value of the parameter λ which would give the best approximation to the actual energies ϵ in equation (47).

For this purpose we take the matrix element in equation (A.8) where $n = n' = 0$ which gives the value

$$F(\lambda, q) \equiv \left[\frac{1}{2}\lambda^2 (l+1) - \lambda \right] - \frac{1}{8}\lambda^2 (2l+2) + \frac{q}{\lambda}(2l+3)(2l+2) \tag{A.10}$$

because

$$b(0l, 0l, 1) = 1 \qquad b(0l, 0l, 2) = (2l+2) \qquad b(0l, 0l, 3) = (2l+3)(2l+2). \tag{A.11}$$

The best value of λ we can choose is the one that minimizes $F(\lambda, q)$ which will be given by

$$\frac{\mathrm{d}F}{\mathrm{d}\lambda} = \lambda(l+1) - 1 - \frac{1}{4}\lambda(2l+2) - \frac{q}{\lambda^2}(2l+3)(2l+2) = 0 \tag{A.12}$$

and it implies the cubic equation

$$\tfrac{1}{2}\lambda^3 (l+1) - \lambda^2 - q(2l+3)(2l+2) = 0. \tag{A.13}$$

For any numerical value of q we have then a corresponding real value $\bar{\lambda} \equiv \lambda(q)$. We shall later consider particular numerical cases. In the matrix of equation (A.8) we have to replace λ by $\bar{\lambda}$ and call it $\bar{\mathbf{H}}^{ql}$, and denote by $\mathbf{N}^l$ the overlap matrix

$$\mathbf{N}^l = \| b(n'l, nl, 2) \|. \tag{A.14}$$

Our variational procedure for determining the energies is through the matrix

$$\bar{\mathbf{H}}^{ql} - \epsilon \mathbf{N}^l \tag{A.15}$$

where ϵ is the value appearing in the Schrödinger equation (47). Thus, the energies ϵ can be determined through the secular equation

$$\det[\bar{\mathbf{H}}^{ql} - \epsilon \mathbf{N}^l] = 0 \tag{A.16}$$

where det stands for the determinant of the matrix indicated.

Equation (A.16) has numerical limitations as the computer capacity of solving algebraic equations of high order is limited. Thus, we turn to an equivalent procedure. For this we note that $\mathbf{N}^l$, being a symmetric real matrix, can be diagonalized and thus we can write

$$\mathbf{N}^l = \tilde{\mathbf{O}}^l \mathbf{D} \mathbf{O}^l \tag{A.17}$$

where $\mathbf{O}^l$ is an orthogonal matrix and $\tilde{\mathbf{O}}^l$ is its transposed matrix, while $\mathbf{D}$ is a diagonal matrix all of whose elements turn out to be positive and different from zero.

Thus equation (A.15) can be replaced by

$$\mathbf{Q}^{ql} - \epsilon \mathbf{I} \qquad \text{where} \quad \mathbf{Q}^{ql} \equiv \mathbf{D}^{-1/2} \mathbf{O}^l \bar{\mathbf{H}}^{ql} \tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \tag{A.18}$$

where $\mathbf{I}$ is the unit matrix of the same dimension as the others. Then we can obtain the values of ϵ by diagonalizing the matrix $\mathbf{Q}^{ql}$ in equation (A.18) for given values of q and l [10]. This ϵ is dimensionless but we can obtain it in terms of GeV by the factor in equation (18), i.e.

$$\epsilon' = \left(\frac{\mu' b'^4}{\hbar^2} \right) \epsilon. \tag{A.19}$$

Having outlined the procedure for obtaining the values of ϵ, we proceed to find the eigenvectors corresponding to the eigenvalues. We note then that the process of diagonalization for $\mathbf{Q}^{ql}$ can be carried out by a matrix $\mathbf{R}$ from which, by a similarity transformation, we get a diagonal matrix $\boldsymbol{\Delta}$, i.e.

$$\mathbf{R}^{-1} \mathbf{Q}^{ql} \mathbf{R} = \boldsymbol{\Delta} \tag{A.20}$$

where $\boldsymbol{\Delta}$ gives the non-relativistic eigenvalues of table 1 and figure 1. Multiplying this equation by $\mathbf{R}$ we see from equation (A.18) that we obtain a system of linear algebraic equations

$$\mathbf{D}^{-1/2} \mathbf{O}^l \bar{\mathbf{H}}^{ql} \tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{R} = \mathbf{R} \boldsymbol{\Delta}. \tag{A.21}$$

Multiplying both sides of this equation by $\tilde{\mathbf{O}}^l \mathbf{D}^{-1/2}$ and using equation (A.17) we obtain

$$(\mathbf{N}^l)^{-1} \bar{\mathbf{H}}^{ql} (\tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{R}) = \tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{R} \boldsymbol{\Delta} \tag{A.22}$$

and multiplying by $\mathbf{N}^l$ we have

$$\bar{\mathbf{H}}^{ql} (\tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{R}) - \mathbf{N}^l \bar{\mathbf{H}}^{ql} (\tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{R}) \boldsymbol{\Delta} = 0. \tag{A.23}$$

Thus, the eigenvectors of $\bar{\mathbf{H}}^{ql}$ are the columns of the matrix $\tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{R}$, where $\tilde{\mathbf{O}}^l$ and $\mathbf{D}$ are obtained from the diagonalization of $\mathbf{N}^l$ while $\mathbf{R}$ is determined though the diagonalization of $\mathbf{Q}^{ql}$ as in equation (A.20).

Once we have the eigenvectors, which we will denote as

$$(\tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{R})_A = \tilde{\mathbf{O}}^l \mathbf{D}^{-1/2} \mathbf{r}_A \equiv \mathbf{c}_A \tag{A.24}$$

corresponding to the eigenvalue ϵ_A, we can pass to the eigenstates with the help of the following column vector

$$\boldsymbol{\alpha} = \begin{bmatrix} |1\rangle \\ |2\rangle \\ \vdots \\ |n\rangle \\ \vdots \\ |N\rangle \end{bmatrix} \tag{A.25}$$

where the kets are a short hand notation for

$$|n+1\rangle = R_{nl}(r) \tag{A.26}$$

and in our analysis the number of kets that we considered was $N = 100$.

The eigenstates corresponding to the eigenvalue ϵ_A take the form

$$\tilde{\mathbf{c}}_A \boldsymbol{\alpha} = \sum_{n=1}^{N} c_{An} |n\rangle. \tag{A.27}$$

These are the eigenstates with respect to which we take the expectation value of the $(1/c^2)$ part of the Hamiltonian in equation (45) to give the values that appear in table 2.

References

[1] Dirac P A M 1947 *The Principles of Quantum Mechanics* 3rd edn (Oxford: Clarendon) pp 252–74
Schiff L I 1968 *Quantum Mechanics* (New York: McGraw Hill) pp 311–4
[2] Barut A O and Strobel G L 1986 *Few-Body Systems* **1** 167
Barut A O and Komy S 1985 *Fortschr. Phys.* **33** 6
[3] Moshinsky M and Loyola G 1993 *Found. Phys.* **23** 197
[4] Carbonell J, Desplanques B, Karmanov V A and Mathiot J F 1998 *Phys. Rep.* **300** 215
[5] Lucha W, Strobel F F and Gromes D 1991 *Phys. Rep.* **200** 127
Bali G 2001 *Phys. Rep.* **343** 1
Eichten E, Gottfried K, Kinoshita T, Lane K D and Yan T M 1978 *Phys. Rev.* D **17** 3090
Brambilla N and Vairo A 1999 Quark confinement and the hadron spectrum *Preprint* hep-ph/9904330
[6] Condon E U and Shortley G H 1953 *The Theory of Atomic Spectra* (Cambridge: Cambridge University Press) p 130
[7] Hagiwara K *et al* 2002 Review of particle physics *Phys. Rev.* D **66** 1
[8] Moshinsky M and Smirnov Yu F 1996 *The Harmonic Oscillator in Modern Physics* (The Netherlands: Harwood Academic) p 195
[9] Riquer V 2002 *PhD Thesis* México
Moshinsky M and Hernández Saldaña H 2000 *Rev. Mex. Fis.* **46** 1
[10] Moshinsky M and Riquer V 2002 *Rev. Mex. Fis.* (Suppl 2) **48** 37